Pivot method for solving system of linear equations and an interesting mistake

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Question

Find the constants h and k such that the system of equations:

$$\begin{cases} x + 2y - 6z = k \\ 2x + hy + z = h \\ 3x + 7y - 5z = 14 \end{cases}$$

has (a) a unique solution

(b) infinitely many solutions.

Solution

We start with the augmented matrix:

$$\begin{pmatrix} \boxed{1} & 2 & -6 & k \\ 2 & h & 1 & h \\ 3 & 7 & -5 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -6 & k \\ 0 & h-4 & 13 & h-2k \\ 0 & 1 & 13 & 14-3k \end{pmatrix}$$

Here:

(1) We choose the element in the first row and first column as a pivot: $\boxed{1}$. (enclosed with square).

(2) The elements of the second row are filled with the determinants:

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & h \end{vmatrix}, \begin{vmatrix} 1 & -6 \\ 2 & 13 \end{vmatrix}, \begin{vmatrix} 1 & k \\ 2 & h \end{vmatrix} \quad \text{respectively.}$$

(3) The third row are filled with the determinants:

$$\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}, \begin{vmatrix} 1 & -6 \\ 3 & -5 \end{vmatrix}, \begin{vmatrix} 1 & k \\ 3 & 14 \end{vmatrix} \quad \text{respectively.}$$

(4) Please check how the determinants in (2) and (3) above are constructed. Note that the pivot ($\boxed{1}$) must stay in all determinants and the first column becomes $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Then we continue:

$$\begin{pmatrix} 1 & 2 & -6 & k \\ 0 & \boxed{h-4} & 13 & h-2k \\ 0 & 1 & 13 & 14-3k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -6 & k \\ 0 & \boxed{h-4} & 13 & h-2k \\ 0 & 0 & 13(h-5) & (h-4)(14-3k)-(h-2k) \end{pmatrix}$$

Here :

(1) The third row are filled with the determinants:

$$\begin{vmatrix} \boxed{h-4} & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} \boxed{h-4} & h-4 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} \boxed{h-4} & 13 \\ 1 & 13 \end{vmatrix}, \begin{vmatrix} \boxed{h-4} & h-2k \\ 1 & 14-3k \end{vmatrix} \quad \text{respectively.}$$

(2) Note that the pivot ($\boxed{h-4}$) must stay in all determinants.

So the system of equations is equivalent to :

$$\begin{cases} x + 2y - 6z = k \\ 0x + (h-4)y + 13z = h - 2k \\ 0x + 0y + 13(h-5)z = (h-4)(14-3k) - (h-2k) \end{cases} \quad \ldots. \quad (1)$$

Therefore :

(a) The system has a unique solution if 13(h – 5) ≠ 0, i.e. h ≠ 5 and k can take any real value.

(b) The system has infinitely many solutions if:

$$\begin{cases} 13(h-5) = 0 \\ (h-4)(14-3k) - (h-2k) = 0 \end{cases}$$

i.e. h = 5 and k = 9 .

Mistake

In the above calculation, if h = 4 , the system (1) becomes :

$$\begin{cases} x + 2y - 6z = k \\ 0x + 0y + 13z = 4 - 2k \\ 0x + 0y - 13z = -(4-2k) \end{cases} \quad \ldots. \quad (2)$$

Note that the last two equations in (2) are equivalent (differ only by a negative sign). So (2) becomes:

$$\begin{cases} x + 2y - 6z = k \\ 0x + 0y + 13z = 4 - 2k \end{cases}$$

Solving we get : $(x, \; y, \; z) = \left(\frac{k+24}{13} - 2t, \; t, \; \frac{4-2k}{13}\right)$, where t is a parameter.

We have infinitely many solutions when h = 4 !

We check the original system when h = 4:

$$\begin{cases} x + 2y - 6z = k \\ 2x + 4y + z = 4 \\ 3x + 7y - 5z = 14 \end{cases}$$

The coefficient determinant $\begin{vmatrix} 1 & 2 & -6 \\ 2 & 4 & 1 \\ 3 & 7 & -5 \end{vmatrix} = -13 \neq 0$, which meant that

the system should NOT have infinitely many solutions. It has a unique solution when h = 4.

The reader is requested to find the mistake before scrolling down!

Correct Way

The matrix calculation should better be done like this: (Pivots are enclosed with squares.)

$$\begin{pmatrix} \boxed{1} & 2 & -6 & k \\ 2 & h & 1 & h \\ 3 & 7 & -5 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -6 & k \\ 0 & h-4 & 13 & h-2k \\ 0 & 1 & 13 & 14-3k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -6 & k \\ 0 & \boxed{1} & 13 & 14-3k \\ 0 & h-4 & 13 & h-2k \end{pmatrix}$$

(the second row interchange with the third row)

$$\sim \begin{pmatrix} 1 & 2 & -6 & k \\ 0 & 1 & 13 & 14-3k \\ 0 & 0 & 13(5-h) & (h-2k)-(h-4)(14-3k) \end{pmatrix}$$

We don't have much problem now if the pivots have no variables.

Therefore :

(a) The system has a unique solution if $13(5-h) \neq 0$, i.e. $h \neq 5$ and k can take any real value.

(b) The system has infinitely many solutions if:

$$\begin{cases} 13(5-h) = 0 \\ (h-2k)-(h-4)(14-3k) = 0 \end{cases}$$

i.e. $h = 5$ and $k = 9$.

Reason for the mistake:

If we choose the variable as pivot, here, h – 4 in this case and when h = 4, we have already multiply an equation in the system by 0 in handling another equation in the system.